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On the Coulomb and higher-order sum rules in the relativistic Fermi gas

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Abstract

Two different methods for establishing a space-like Coulomb sum rule for the relativistic Fermi gas are compared. Both of them divide the charge response by a normalizing factor such that the reduced response thus obtained fulfills the sum rule at large momentum transfer. To determine the factor, in the first approach one exploits the scaling property of the longitudinal response function, while in the second one enforces the completeness of the states in the space-like domain via the Foldy-Wouthuysen transformation. The energy-weighted and the squared-energy-weighted sum rules for the reduced responses are explored as well and the extension to momentum distributions that are more general than a step-function is also considered. The two methods yield reduced responses and Coulomb sum rules that saturate in the non-Pauli-blocked region, which can hardly be distinguished for Fermi momenta appropriate to atomic nuclei. Notably the sum rule obtained in the Foldy-Wouthuysen approach coincides with the well known non-relativistic

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one. Only at quite large momentum transfers (say 1 GeV/c) does a modest softening of the Foldy-Wouthuysen reduced response with respect to that obtained in the scaling framework show up. The two responses have the same half-width to second order in the Fermi momentum expansion. However, when distributions extending to momenta larger than that at the Fermi surface are employed, then in both methods the Coulomb sum rule saturates only if the normalizing factors are appropriately modified to account for the high momentum components of the nucleons.

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I. INTRODUCTION

In this paper we compare two different approaches to the relativistic Coulomb sum rule confining ourselves to dealing with a homogeneous, translationally invariant system of non-interacting nucleons, namely the Fermi gas (FG). We consider not only the customary step-function, but more general momentum distributions as well in order to reflect some of the correlations among the nucleons. Our chief aim is to explore whether the relativistic Coulomb sum rule displays the same basic feature as the non-relativistic one, i.e. saturation at large transferred momentum. Indeed, as is well-known, for a FG with Z non-interacting protons the non-relativistic Coulomb sum rule (NRCSR) reads (k_F is the Fermi momentum)

$$\int_0^\infty \frac{R_L(\boldsymbol{q},\omega)}{ZG_E^2(Q^2)}d\omega = \int_0^\infty r_L(\boldsymbol{q},\omega)d\omega = \Sigma_C^{nr}(q)$$
$$= \left[\vartheta(q-2k_F) + \frac{3q}{4k_F}\left(1-\frac{q^2}{12k_F^2}\right)\vartheta(2k_F-q)\right] \tag{1.1}$$

and exhibits saturation as a consequence of unitarity (summation over a complete set of states) in the non-Pauli-blocked regime. Here the NRCSR just counts the charged particles inside the system. In (1.1) R_L is the usual longitudinal response function whereas r_L is commonly referred to as the reduced longitudinal response function, since its dependence upon the physics of the nucleon has been divided out ($G_E(Q^2)$ is the electric form factor, here of the proton). Note that the implementation of unitarity in the non-relativistic regime requires extending the range of integration in (1.1) up to infinity.

Now from the well-known symmetry property of R_L in the Pauli-blocked region [1,2], namely

$$R_L^b(q,\omega) = R_L^{nb}(q,\omega) - R_L^{nb}(q,-\omega) \quad , \tag{1.2}$$

the relationship

$$\int_0^\infty \frac{R_L^{nb}(\boldsymbol{q},\omega)}{ZG_E^2(Q^2)}d\omega = \int_0^\infty r_L^{nb}(q,\omega)d\omega = \frac{1}{2}\left[1+\Sigma_C^{nr}(q)\right] \tag{1.3}$$

may be deduced (the superscript b(nb) stands for Pauli(non-Pauli)-blocked), which allows one to express the NRCSR *solely* in terms of the non-Pauli-blocked reduced response over the whole range of q.

When attempting to generalize the NRCSR to a relativistic homogeneous system with an equal number of non-interacting protons and neutrons (the symmetric relativistic Fermi gas (RFG)) one faces at least two issues in order to achieve saturation: first the neutrons and protons composite nature, which in the relativistic domain is not so straightforward to factor out, should be accounted for; moreover the closure relation is no longer restricted to particle-hole (ph) excitations, but includes the time-like region (particle-antiparticle ($p\overline{p}$) excitations) as well.

For an ideal system of pointlike non-interacting nucleons without anomalous magnetic moments the first of the above items is of course avoided, whereas the second, as shown by Walecka [4] and Matsui [5], can be dealt with without difficulty: a relativistic Coulomb

sum rule (RCSR) that is quite close to the NRCSR is thus obtained for densities of the RFG roughly corresponding to those in nuclei by exploiting closure in both space-like and time-like regions, i.e. by integrating over the whole range of positive energies (of course for a given q the actual range of integration is cut by the sharp boundaries of the FG response function).

However experimentally the structure of the nucleon should be reckoned with and the time-like energy domain cannot be reached with the presently available experimental facilities. Therefore one would like to define a sum rule where the physics of the nucleon has been disentangled (at least to a large extent) and saturation obtains in the space-like energy regime alone. For this purpose a method referred to as the *scaling approach* has been developed by Alberico et al. [6]. In the present paper another one, based on the Foldy-Wouthuysen (FW) transformation, is suggested and compared with the scaling method not only in connection with the Coulomb sum rule, but also with those having energy and energy squared weightings.

II.

A. The scaling approach

It is well-known that in the non-relativistic FG for large enough transferred momenta, namely for $q > 2k_F$, the reduced response r_L scales, i.e. it becomes function of only one variable, the so-called scaling variable, which corresponds to the minimum momentum parallel or antiparallel to $\boldsymbol{q}$ that a nucleon inside the system can have in order to contribute to the response. A dimensionless non-relativistic scaling variable can indeed be defined as [3][1]

$$\psi_{nr} \equiv \frac{k_\parallel}{m_N} = \frac{m_N}{k_F}\left(\frac{\omega}{q} - \frac{q}{2m_N}\right) = \frac{1}{\eta_F}\left(\frac{\lambda}{\kappa} - \kappa\right) \quad , \tag{2.1}$$

where m_N is the nucleon mass, in terms of which the non-relativistic reduced longitudinal response reads

$$\begin{aligned} r_L^{nrFG}(\kappa,\lambda) = \frac{3}{4m_N\kappa\eta_F^3}\Bigg\{&\vartheta\left(\eta_F - \kappa\right)\left[\frac{\eta_F^2}{2}(1-\psi_{nr}^2)\vartheta(1-\psi_{nr})\vartheta\left(\frac{\kappa}{\eta_F} - \frac{1-\psi_{nr}}{2}\right) + \right.\\ &\left. + 2\lambda\vartheta\left(\frac{1-\psi_{nr}}{2} - \frac{\kappa}{\eta_F}\right)\right] + \frac{\eta_F^2}{2}(1-\psi_{nr}^2)\vartheta(1-\psi_{nr}^2)\vartheta\left(\kappa - \eta_F\right)\Bigg\} \quad . \end{aligned} \tag{2.2}$$

Thus, as an alternative to (1.1), one can obtain the NRCSR as an integral over the scaling variable. Indeed it is easily verified that

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[1]Following ref. [7] we introduce here the dimensionless momentum and energy transfer to the nucleus, the momentum and energy of a nucleon inside the RFG and the momentum and Fermi energy according to:
$\kappa \equiv q/2m_N$, $\lambda \equiv \omega/2m_N$ $\eta \equiv k/m_N$, $\epsilon \equiv \sqrt{1+\eta^2}$, $\eta_F \equiv k_F/m_N$, $\epsilon_F \equiv \sqrt{1+\eta_F^2}$.
Moreover $\tau = \kappa^2 - \lambda^2$ and $\xi_F \equiv \epsilon_F - 1$.

$$\int_{-1}^{+1} 2m_N r_L^{nrFG}(\kappa,\lambda)\frac{\partial\lambda}{\partial\psi_{nr}}d\psi_{nr} = \Sigma_C^{nr}(\kappa) \quad . \tag{2.3}$$

Also for the RFG, a scaling variable can be defined, again representing the minimum longitudinal momentum of the nucleon absorbing the virtual photon inside nuclear matter. It reads [7]

$$\psi \equiv \sqrt{\frac{\gamma_- - 1}{\xi_F}}\left[\vartheta(\lambda-\lambda_0)-\vartheta(\lambda_0-\lambda)\right] , \tag{2.4}$$

where $\lambda_0 = \frac{1}{2}\left[\sqrt{1+4\kappa^2}-1\right]$ is the dimensionless quasielastic peak energy and $\gamma_- \equiv \kappa\sqrt{1+\frac{1}{\tau}}-\lambda$. In terms of the above the space-like longitudinal response function is then expressed as follows [7]

$$\begin{aligned} R_L^{RFG}(\kappa,\lambda) &= \frac{3\mathcal{N}}{4m_N\kappa\eta_F^3}\left(\epsilon_F-\Gamma\right)\vartheta\left(\epsilon_F-\Gamma\right)U_L(\kappa,\lambda) \\ &= \frac{H_L(\kappa,\lambda)}{2m_N}\frac{\partial\psi}{\partial\lambda}\frac{3}{4}\vartheta\left(\epsilon_F-\Gamma\right)\left\{\vartheta\left(\eta_F-\kappa\right)\left[(1-\psi^2)\vartheta\left(\kappa\sqrt{1+\frac{1}{\tau}}+\lambda-\epsilon_F\right)+\right.\right. \\ &\left.\left.+\frac{2\lambda}{\xi_F}\vartheta\left(\epsilon_F-\kappa\sqrt{1+\frac{1}{\tau}}-\lambda\right)\right]+\vartheta\left(\kappa-\eta_F\right)\left(1-\psi^2\right)\right\} \quad , \end{aligned} \tag{2.5}$$

and is non-vanishing in the range $\lambda_{min} < \lambda < \lambda_{max}$, where

$$\lambda_{max,min} \equiv \frac{1}{2}\left[\sqrt{1+(2\kappa\pm\eta_F)^2}-\epsilon_F\right] \quad . \tag{2.6}$$

In (2.5)

$$U_L(\kappa,\lambda) = \frac{\kappa^2}{\tau}\left[G_E^2(\tau)+W_2(\tau)\Delta\right] \tag{2.7}$$

and

$$H_L^{RFG}(\kappa,\lambda) = \frac{2\xi_F}{\kappa\eta_F^3}\frac{\mathcal{N}}{\frac{\partial\psi}{\partial\lambda}}U_L(\kappa,\lambda) \equiv \frac{2\xi_F}{\kappa\eta_F^3}\frac{\mathcal{N}}{\frac{\partial\psi}{\partial\lambda}}\frac{\kappa^2}{\tau}\left[G_E^2(\tau)+W_2(\tau)\Delta\right] \tag{2.8}$$

with $W_2(\tau) = \frac{1}{1+\tau}\left(G_E^2(\tau)+\tau G_M^2(\tau)\right)$, $G_M(\tau)$ being the magnetic form factor of the nucleon. Furthermore

$$\begin{aligned} \Delta \equiv \frac{\langle k_\perp^2\rangle}{m_N^2} &= \frac{1}{\epsilon_F-\Gamma}\int_\Gamma^{\epsilon_F} d\epsilon\eta_\perp^2(\epsilon,\kappa,\lambda) = \\ &= \frac{\tau}{\kappa^2}\left[\frac{(\epsilon_F^2+\Gamma\epsilon_F+\Gamma^2)}{3}+\lambda(\epsilon_F+\Gamma)+\lambda^2\right]-(1+\tau)\,, \end{aligned} \tag{2.9}$$

where $\Gamma \equiv \max\left[\epsilon_F-2\lambda,\gamma_-\right]$ and $\eta_\perp^2(\epsilon,\kappa,\lambda) = \frac{\tau}{\kappa^2}(\epsilon+\lambda)^2-(1+\tau)$ corresponds to the average quadratic transverse momentum in units of m_N^2. Indeed the motion of the nucleons transverse to $\boldsymbol{q}$ introduces a *magnetic* contribution into the *charge* response [8].

It should be understood that the charge response is calculated by adding the contribution with neutrons where $\mathcal{N} = N$ and with protons where $\mathcal{N} = Z$.

The structure of (2.5) naturally suggests introducing a reduced response according to

$$r_L^{RFG}(q,\omega) = \frac{R_L^{RFG}(q,\omega)}{H_L^{RFG}(q,\omega)} \quad . \tag{2.10}$$

For $\kappa > \eta_F$, where Pauli correlations are no longer effective, this reads

$$r_L(q,\omega) = \frac{3}{4}\frac{1}{2m_N}\left(1-\psi^2\right)\vartheta\left(1-\psi^2\right)\frac{\partial\psi}{\partial\lambda} = S(\psi)\frac{\partial\psi}{\partial\lambda} \tag{2.11}$$

and is indeed well suited to be integrated over ψ, since it scales[2] and has the Jacobian incorporated.

The RCSR for the reduced longitudinal response in the non-Pauli-blocked region ($\kappa > \eta_F$) is thus easily obtained [9] according to

$$\Sigma_C(q) = \int_0^q d\omega \frac{R_L^{RFG}(q,\omega)}{H_L^{RFG}(\kappa,\lambda)} = \int_0^\kappa d\lambda \frac{d\psi}{d\lambda} S(\psi) = \int_{-1}^{+1} d\psi S(\psi) = 1 \quad , \tag{2.12}$$

where the range of integration is cut on the light front, since no anti-nucleon physics is contained in our R_L: therefore $\Sigma_C(q)$ is directly accessible to the experiment. Because of the scaling it thus appears that it is possible to define for the RFG a *space-like* reduced response fulfilling a *space-like* Coulomb sum rule which saturates when the Pauli correlations vanish.

Although the RFG is not trustworthy as a model for nuclei at small q (here the surface matters), one would still like to get a complete analytical expression for $\Sigma_C(q)$ as obtained via the scaling method in the RFG. This turns out to be extremely cumbersome for $q \leq 2k_F$. However (1.3), although no longer "exactly" valid, still provides an approximate expression for the RFG Coulomb sum rule in the Pauli-blocked domain, which is quite accurate in the range of densities appropriate for nuclei (see Fig. 3). It reads

$$\Sigma_C(q) \approx \left[2\Sigma_C(q)^{nb} - 1\right] = \frac{3}{2}\sqrt{\frac{\sqrt{1+\kappa^2}-1}{\xi_F}}\left[1 - \frac{1}{3\xi_F}\left(\sqrt{1+\kappa^2}-1\right)\right] \tag{2.13}$$

and reduces to the non-relativistic Pauli-blocked Coulomb sum rule Σ_C^{nr} in the small momentum, small density limit.

B. The Foldy-Wouthuysen approach

We shall now attempt to recover the saturation value for the Coulomb sum rule by exploiting the FW formalism, which has been previously successfully used in a potential

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[2]In fact $S(\psi)$ is proportional to the scaling function defined in [7]

model description of nuclear matter [10]. The FW framework yields an expression for the response function which is convenient for the non-relativistic reduction, although it does not transparently display covariance.

Through the unitary transformation

$$\mathcal{T}(\boldsymbol{k}) = \sqrt{\frac{E_k + m_N}{2E_k}} \left(1\!\!1 + \frac{\boldsymbol{\gamma} \cdot \boldsymbol{k}}{E_k + m_N} \right) \tag{2.14}$$

the FW reduced Green's function is defined as follows

$$G^{\mathrm{FW}}(k) = \mathcal{T}(\boldsymbol{k}) G(k) \mathcal{T}(\boldsymbol{k}) \quad , \tag{2.15}$$

where, in the non-interacting case [11,12],

$$G(k) = G_0(k) = \frac{(\not k + m)}{2E_k} \left[\frac{\vartheta(k - k_F)}{k_0 - E_k + i\varepsilon} + \frac{\vartheta(k_F - k)}{k_0 - E_k - i\varepsilon} - \frac{1}{k_0 + E_k - i\varepsilon} \right] \tag{2.16}$$

with $E_k = \sqrt{k^2 + m_N^2}$, i.e. the free relativistic energy. Owing to the identity

$$\mathcal{T}(\boldsymbol{k}) \left(\gamma^\mu k_\mu + m \right) \mathcal{T}(\boldsymbol{k}) = P_+ (k_0 + E_k) - P_- (k_0 - E_k) \tag{2.17}$$

one then gets

$$G_0^{\mathrm{FW}}(k) = P_+ \left[\frac{\vartheta(k - k_F)}{k_0 - E_k + i\varepsilon} + \frac{\vartheta(k_F - k)}{k_0 - E_k - i\varepsilon} \right] - P_- \frac{1}{k_0 + E_k - i\varepsilon} \ , \tag{2.18}$$

where the operators $P_\pm = \frac{1\!\!1 \pm \gamma^0}{2}$ project on the large/small components of the wave function.

Fig. 1 illustrates the passage from the Feynman to the FW rules: clearly both the vertices and the Green's functions are changed. The latter are given by (2.18) and the former by

$$\Gamma^\mu_{FW}(q) \equiv \mathcal{T}^\dagger(\boldsymbol{k} + \boldsymbol{q}) \Gamma^\mu(q) \mathcal{T}^\dagger(\boldsymbol{k}) \tag{2.19a}$$
$$\tilde{\Gamma}^\mu_{FW}(-q) \equiv \mathcal{T}^\dagger(\boldsymbol{k}) \Gamma^\mu(-q) \mathcal{T}^\dagger(\boldsymbol{k} + \boldsymbol{q}) \quad , \tag{2.19b}$$

where $\Gamma^\mu(q) = F_1(Q^2)\gamma^\mu + i\frac{F_2(Q^2)}{2m_N}\sigma^{\mu\nu} q^\nu$.

The leading approximation to the space-like charge response amounts to ignoring all momentum dependence or, equivalently, to allowing for a very large m_N in the vertices (2.19). In this scheme one obtains the following FW longitudinal response function

$$\begin{aligned} \mathcal{R}_L^{FW}(q,\omega) &= -\frac{V}{\pi} \mathrm{Im}\Pi_{FW}(q,\omega) = \frac{V}{\pi} \mathrm{Im} \left\{ i \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr} \left[P_+ G_0^{\mathrm{FW}}(k) P_+ G_0^{\mathrm{FW}}(k+q) \right] \right\} \\ &= \frac{3\mathcal{N}}{4m_N \kappa \eta_F^3} \left(\epsilon_F - \Gamma \right) \vartheta \left(\epsilon_F - \Gamma \right) \left[\frac{\kappa^2}{\tau} \left(1 + \tau + \Delta \right) - \lambda^2 \right] \quad , \end{aligned} \tag{2.20}$$

where the polarization propagator $\Pi_{FW}(q,\omega)$, calculated in terms of the pointlike vertices, has been introduced and where V is the (large) volume enclosing the system.

Now, since the only nonzero contributions to Π_{FW} come from ph excitations (space-like region), while those arising from $p\overline{p}$ excitations are projected out, when integrating

$\mathcal{R}_{FW}(q,\omega)$ over the energy ω the closure relation *already applies in the space-like region*, yielding a sum rule which notably coincides with the NRCSR (1.1). It is of importance to realize that as soon as the zero-momentum approximation in the vertices is removed, then one goes beyond the space-like domain and time-like contributions come into play. Also of relevance is the fact that (2.20) cannot be expressed in terms of the RFG scaling variable (2.4).

The complete longitudinal response in the RFG model is expressed in the FW framework as

$$\begin{aligned} R_L^{FW}(q,\omega) &= -\frac{V}{\pi}\mathrm{Im}\Pi^{00}(q,\omega) = V\int_0^{k_F}\frac{d^3k}{(2\pi)^3}\delta(\omega+E_{\boldsymbol{k}}-E_{\boldsymbol{k}+\boldsymbol{q}})\times \\ &\times\vartheta(|\boldsymbol{k}+\boldsymbol{q}|-k_F)\mathrm{Tr}\left[P_+\Gamma^0_{FW}(q)P_+\tilde{\Gamma}^0_{FW}(-q)\right] \\ &= \mathcal{R}_L^{FW}(q,\omega)\frac{\frac{\kappa^2}{\tau}\left[G_E^2(\tau)+\Delta W_2(\tau)\right]}{\frac{\kappa^2}{\tau}\left[1+\tau+\Delta\right]-\lambda^2}\quad, \end{aligned} \tag{2.21}$$

which, of course, coincides with (2.5). Likewise in the scaling scheme, it is then natural to introduce in the FW framework a reduced response according to

$$r_L^{FW}(\kappa,\lambda) = \frac{R_L^{FW}(\kappa,\lambda)}{H_L^{FW}(\kappa,\lambda)} \equiv \frac{1}{\mathcal{N}}\mathcal{R}_L^{FW}(\kappa,\lambda)\quad, \tag{2.22}$$

with the normalizing factor

$$H_L^{FW}(\kappa,\lambda) \equiv \mathcal{N}\frac{\frac{\kappa^2}{\tau}\left[G_E^2(\tau)+\Delta W_2(\tau)\right]}{\frac{\kappa^2}{\tau}\left[1+\tau+\Delta\right]-\lambda^2}\quad. \tag{2.23}$$

The $r_L^{FW}(\kappa,\lambda)$ defined here then leads automatically to a RCSR which coincides with the NRCSR. Thus the square roots expressing the energy in the relativistic propagator (2.18) alter the response with respect to the non-relativistic case, but leave the area under the latter unchanged.

C. Comparing the two methods

We now compare $H_L^{FW}(\kappa,\lambda)$ with $H_L^{RFG}(\kappa,\lambda)$ in the low-density regime which applies to real nuclei. This is done in Fig. 2 where the two reducing factors are displayed as a function of ω for $q = 500$ MeV/c (panel A) and $q = 1$ GeV/c (panel B) respectively. One sees in the figure that close to the peak the positive H_L^{RFG} and H_L^{FW} essentially coincide, whereas at low (high) frequencies H_L^{RFG} is larger (smaller) than H_L^{FW}. It is thus clear why both the scaling and FW reduced responses yield the same RCSR in the non-Pauli-blocked regime: indeed they practically coincide at the peak and their contributions to the sum rule arising from the edges of the response region add up to the same amount. This compensation, while almost perfect in the non-Pauli-blocked domain, is not complete in the Pauli-blocked one. However the difference is very small, as is apparent in fig. 3 where the sum rule obtained through the scaling approach is plotted at two different Fermi momenta. The saturation in the non-Pauli-blocked region is evident and, moreover, in the Pauli-blocked one, the scaling

sum rule is almost indistinguishable from the NRCSR at normal density. Importantly, even at very large densities, the difference between the two remains quite small.

Let us now compare the reducing factors in leading order of the η_F expansion. For this purpose we recall following [13] that

$$\frac{\partial\psi}{\partial\lambda} = \frac{\kappa}{\tau}\frac{\sqrt{1+\xi_F\psi^2/2}}{\sqrt{2\xi_F}}\left[\frac{1+2\lambda+\xi_F\psi^2}{1+\lambda+\xi_F\psi^2}\right] \tag{2.24}$$

$$= \frac{\kappa}{\eta_F\tau}\left(\frac{1+2\lambda}{1+\lambda}\right) + O\left[\eta_F^2\right] \quad . \tag{2.25}$$

Therefore one obtains

$$H_L^{RFG}(\kappa,\lambda) = \frac{1+\lambda}{1+2\lambda}\mathcal{N}G_E^2(\tau) + \mathcal{O}[\eta_F^2] \tag{2.26}$$

$$= \frac{1+\tau}{1+2\tau}\left[1 - \frac{1}{1+2\tau}\sqrt{\frac{\tau}{1+\tau}}\eta_F\psi\right]\mathcal{N}G_E^2(\tau) + \mathcal{O}[\eta_F^2] \tag{2.27}$$

and

$$H_L^{FW}(\kappa,\lambda) = \frac{1}{1+\frac{\tau^2}{\kappa^2}}\mathcal{N}G_E^2(\tau) + \mathcal{O}[\eta_F^2] \tag{2.28}$$

$$= \frac{1+\tau}{1+2\tau}\left[1 + \frac{2\tau}{1+2\tau}\sqrt{\frac{\tau}{1+\tau}}\eta_F\psi\right]\mathcal{N}G_E^2 + (\tau)\mathcal{O}[\eta_F^2] \quad . \tag{2.29}$$

to be compared with the De Forest expression [14]

$$H_L^{\mathrm{DeF}}(\kappa,\lambda) = \frac{1+\tau}{1+2\tau}\mathcal{N}G_E^2(\tau) \quad . \tag{2.30}$$

One thus sees in the low-density regime where $\eta_F \ll 1$ that H_L^{RFG}, H_L^{FW} and H_L^{DeF} all coalesce at the peak of the RFG response, where $\psi = 0$, $\lambda = \tau$ and $\kappa^2 = \tau(\tau+1)$. Note also that the differences of $\mathcal{O}[\eta_F]$ are linear in ψ and so tend to cancel when forming integrals over the scaling variable with integrands that are symmetric around the quasielastic peak.

III. THE TIME-LIKE REGION

As shown by Matsui [5], the space-like relativistic Coulomb sum rule for pointlike Dirac particles without anomalous magnetic moments goes to $Z/2$ for large q, showing that for pointlike nucleons on their mass-shell the particle-antiparticle symmetry is reflected in the sharing of the Coulomb sum rule at very large momentum transfers.

Here we consider contributions to the time-like longitudinal response function arising from the RFG. Thus we write, using the $p\overline{p}$ polarization propagator,

$$R_L^{tl}(\kappa,\lambda) \equiv -\frac{V}{\pi}\mathrm{Im}\left[\Pi^{00}_{p\overline{p}}(\kappa,\lambda)|_{k_F=0} - \Pi^{00}_{p\overline{p}}(\kappa,\lambda)|\right] =$$

$$= \frac{3\mathcal{N}}{4m_N\kappa\eta_F^3}\left(\tilde{\Gamma}-\zeta_-\right)\vartheta\left(\tilde{\Gamma}-\zeta_-\right)U_L^{tl}(\kappa,\lambda) \quad , \tag{3.1}$$

where, in terms of the time-like electric and magnetic form factors (recall that the magnitude of the negative τ always exceeds one in the response region),

$$U_L^{tl}(\kappa,\lambda) \equiv \frac{\kappa^2}{|\tau|}\left[G_{E,t}^2(\tau) + W_2(\tau)\Delta_{p\overline{p}}\right] \quad , \tag{3.2}$$

$$W_2(\tau) = \frac{1}{1+\tau}\left[G_{E,t}^2(\tau) + \tau G_{M,t}^2(\tau)\right] \quad , \tag{3.3}$$

$$\Delta_{p\overline{p}} \equiv \frac{\tau}{\kappa^2}\left[\frac{\left(\zeta_-^2 + \tilde{\Gamma}\zeta_- + \tilde{\Gamma}^2\right)}{3} - \lambda(\zeta_- + \tilde{\Gamma}) + \lambda^2\right] - (1+\tau) \quad , \tag{3.4}$$

$$\zeta_\pm \equiv \lambda \pm \kappa\sqrt{1 - \frac{1}{|\tau|}} \tag{3.5}$$

$$\tag{3.6}$$

and

$$\tilde{\Gamma} \equiv min\left[\epsilon_F, \zeta_+\right] \ . \tag{3.7}$$

The subtraction in the first line of (3.1) also ensures that the divergences stemming from the Dirac sea are canceled. As a consequence the integration of $R_L^{tl}(q,\omega)$ over the time-like region yields the finite contribution [5]

$$\int_q^\infty \frac{1}{Z} R_L^{tl}(q,\omega)d\omega = -\frac{2}{Z}\sum_{|\boldsymbol{k}|<k_F} \frac{\left(E_{\boldsymbol{k}+\boldsymbol{q}} - E_{\boldsymbol{k}}\right)^2 - \boldsymbol{q}^2}{4E_{\boldsymbol{k}}E_{\boldsymbol{k}+\boldsymbol{q}}} \quad , \tag{3.8}$$

which, while vanishing at zero momentum transfer, goes to 1/2 in the large-q limit.

This result, while of theoretical interest, cannot presently be tested against experimental data. Indeed, while in the space-like region R_L^{RFG} qualitatively, although not quantitatively, accounts for the experiments, these are lacking in the time-like domain. Furthermore, for the large energy transfers typical of the time-like sector, the possibility of exciting the nucleon in the scattering process becomes dominant. Finally, even allowing for the possibility of disentangling inelastic nucleonic processes, our knowledge of the time-like elastic nucleonic form factors is much poorer than in the space-like domain.

Yet, to provide a feeling for the time-like physics in the simple point-nucleon RFG model, in Fig. 4 the space-like and time-like longitudinal response functions are plotted as functions of λ for $q = 1000$ MeV/c and $k_F = 250$ MeV/c. Note that the boundaries of the time-like response are the same as the space-like ones but for a shift of ϵ_F (corresponding in ω essentially to $2m_N$). Accordingly the time-like response R_L^{tl} is shown in the figure for a range displaced by the Fermi energy to make the comparison with R_L easier. Note that the off-set of the maximum of R_L^{tl} with respect to that of R_L is no longer given by λ_0 and that in general the two responses have different shapes and norms.

Likewise for purpose of illustration, we display in Fig. 5 the Coulomb sum rule for the RFG of Dirac nucleons, including the contribution stemming from $p\overline{p}$ excitations. The sum rule, defined through a frequency integral spanning both the space-like and the time-like regions and suitably renormalized, reads [5]

$$\Sigma_C(q) = 1 - \frac{3}{4\pi k_F^3}\int d^3k \frac{(E_{\boldsymbol{k}+\boldsymbol{q}} + E_{\boldsymbol{k}})^2 - \boldsymbol{q}^2}{4E_{\boldsymbol{k}}E_{\boldsymbol{k}+\boldsymbol{q}}}\vartheta\left(k_F - k\right)\vartheta\left(k_F - |\boldsymbol{k}+\boldsymbol{q}|\right) \quad . \tag{3.9}$$

One sees from the figure that Σ_C saturates when the Pauli correlations are no longer operative (i.e. for $\kappa > 0.27$ at $k_F = 250$ MeV/c and for $\kappa > 1.06$ at $k_F = 1000$ MeV/c) and that, in the Pauli-blocked region, the difference between the relativistic and the non-relativistic Coulomb sum rule is barely perceptible except for very large densities, since (3.9) correctly reduces to the NRCSR in the limit $k_F < m_N$. Moreover the $p\overline{p}$ contribution starts to be substantially felt only at large k_F.

IV. BEYOND THE STEP-FUNCTION MOMENTUM DISTRIBUTION

With the idea of exploring the impact of nucleon-nucleon correlations on the Coulomb sum rule let us insert into the RFG model momentum distributions that are more general than the step-function. For this purpose, following [15] and exploiting (1.2), we first observe that R_L can be expressed as follows[3]

$$\begin{aligned}
R_L^{RFG}(\kappa,\lambda) &= \frac{1}{4\pi m_N}\left\{\int d^3\eta n(\eta)\frac{X_L^{RFG}(\eta,\kappa,\lambda)}{X_L^{FW}(\eta,\kappa,\lambda)}\delta\left(2\lambda + \sqrt{1+\eta^2} - \sqrt{1+(\boldsymbol{\eta}+2\boldsymbol{\kappa})^2}\right) - \right. \\
&\quad \left. - \int d^3\eta n(\eta)\frac{X_L^{RFG}(\eta,\kappa,-\lambda)}{X_L^{FW}(\eta,\kappa,-\lambda)}\delta\left(-2\lambda + \sqrt{1+\eta^2} - \sqrt{1+(\boldsymbol{\eta}+2\boldsymbol{\kappa})^2}\right)\right\} & (4.1) \\
&= \frac{1}{2m_N}\left\{\int_{y_-}^{\infty} d\eta n(\eta) X_L^{RFG}(\eta,\kappa,\lambda) - \int_{y_+}^{\infty} d\eta n(\eta) X_L^{RFG}(\eta,\kappa,-\lambda)\right\} \quad , & (4.2)
\end{aligned}$$

where

$$y_\pm \equiv |\kappa \pm \lambda\sqrt{1+1/\tau}| \quad ,$$

$$\begin{aligned}
X_L^{RFG}(\eta,\kappa,\lambda) &\equiv \frac{\eta}{2\kappa\epsilon}\left\{(\epsilon+\lambda)^2\frac{1}{1+\tau}\left[G_E^2(\tau) + \tau G_M^2(\tau)\right] - \kappa^2 G_M^2(\tau)\right\} = \\
&= \frac{\eta}{2\kappa\epsilon}\frac{\kappa^2}{\tau}\left(G_E^2(\tau) + W_2(\tau)\eta_\perp^2(\lambda,\kappa)\right) \quad , & (4.3) \\
X_L^{FW}(\eta,\kappa,\lambda) &= \frac{\eta}{2\kappa}(\epsilon + 2\lambda) & (4.4)
\end{aligned}$$

and the momentum distribution $n(\eta)$ is normalized according to

$$\int_0^\infty \eta^2 n(\eta) d\eta = \mathcal{N} \tag{4.5}$$

---

[3] In this expression the energy of the struck nucleon is simply assumed to be the RFG energy $E_{\boldsymbol{p}}$. Thus off-shell effects in the e.m. vertices, which are not easy to predict since we lack a fundamental theory, have been ignored.

separately for protons and neutrons. In the same limit as in (2.20) $X_L \to X_L^{FW}$, which explains the notation of the LHS of (4.4). The response one obtains is then just a generalization of the FW one, namely

$$\mathcal{R}_{FW}(\kappa,\lambda) = \frac{1}{2m_N\mathcal{N}}\left\{\int_{y_-}^{\infty} d\eta n(\eta) X_L^{FW}(\eta,\kappa,\lambda) - \int_{y_+}^{\infty} d\eta n(\eta) X_L^{FW}(\eta,\kappa,-\lambda)\right\} \quad , \tag{4.6}$$

which, setting $n(\eta) = \frac{3\mathcal{N}}{\eta_F^3}\theta(\eta_F - \eta)$, yields (2.20).

For the above response the easily obtained Coulomb sum rule (the tilde is to remind us that a generic momentum distribution is employed) reads

$$\begin{aligned}\tilde{\Sigma}_C \equiv \int_0^q \mathcal{R}_{FW}(q,\omega) d\omega &= \frac{1}{4\pi\mathcal{N}}\int d^3\eta n(\eta)\left[\vartheta\left(\cos\theta + \frac{\kappa}{\eta}\right) - \vartheta\left(-\cos\theta - \frac{\kappa}{\eta}\right)\right] = \\ &= 1 + \frac{1}{\mathcal{N}}\int_\kappa^\infty d\eta\eta^2 n(\eta)\left(\frac{\kappa}{\eta} - 1\right) \quad ,\end{aligned} \tag{4.7}$$

which is bound to saturate in the large momentum limit but for pathological momentum distributions. By expanding (4.7) around $\kappa = 0$ one obtains

$$\tilde{\Sigma}_C \approx \frac{\kappa}{\mathcal{N}}\int_0^\infty d\eta\eta n(\eta) - \frac{n(0)}{6\mathcal{N}}\kappa^3 + O(\kappa^4) \quad , \tag{4.8}$$

which has the behaviour typical of a sum rule of an infinite system.

From (4.8) the NRCSR for the FG is recovered by employing a step-function momentum distribution. On the other hand for a general $n(\eta)$, one can still have the term linear in κ in (4.8) identical to the one appearing in the NRCSR for the FG by setting

$$\eta_F = \frac{3}{2}\frac{\mathcal{N}}{\int_0^\infty d\eta\eta n(\eta)} \quad , \tag{4.9}$$

which can be exploited for relating η_F to a finite nucleus momentum distribution. In this connection we recall that in ref. [9] two procedures have been suggested to determine the Fermi momentum in a way that brings the RFG as close as possible to a real nucleus, one related to the nuclear momentum distribution and the other to the half width of the longitudinal response. These yield for ^{16}O the values $k_F = 1.039$ fm^{-1} and $k_F = 1.22$ fm^{-1} respectively. Here (4.9) yields $k_F = 1.1$ fm^{-1} for ^{16}O, which lies in between the above quoted values.

To define a reducing factor in the presence of a generic momentum distribution it helps to notice that Δ, the quadratic transverse momentum distribution, for any $n(\eta)$ would read

$$\tilde{\Delta} = \frac{\int d^3\eta n(\eta)\frac{1}{\epsilon(\epsilon+2\lambda)}\eta_\perp^2(\kappa,\lambda)}{\int d^3\eta n(\eta)\frac{1}{\epsilon(\epsilon+2\lambda)}} = \frac{\int_{y_-}^\infty d\eta\frac{\eta}{\epsilon}\eta_\perp^2(\kappa,\lambda)n(\eta)}{\int_{y_-}^\infty d\eta\frac{\eta}{\epsilon}n(\eta)} \quad , \tag{4.10}$$

and would reduce to (2.9) for a step-function momentum distribution. Accordingly, in a FW-inspired framework, one might introduce

$$\tilde{r}_L^{FW}(\kappa,\lambda) \equiv \frac{R_L(\kappa,\lambda)}{\tilde{H}_L^{FW}(\kappa,\lambda)} \quad , \tag{4.11}$$

with

$$\tilde{H}_L^{FW}(\kappa,\lambda) = \mathcal{N}\frac{\frac{\kappa^2}{\tau}\left[G_E^2(\tau)+\tilde{\Delta}W_2(\tau)\right]}{\frac{\kappa^2}{\tau}\left[1+\tau+\tilde{\Delta}\right]-\lambda^2} \tag{4.12}$$

and likewise, in the scaling scheme, one would use $\tilde{H}_L^{RFG}(\kappa,\lambda)$ as given by (2.8), but with $\tilde{\Delta}$ replacing Δ.

To get a feeling for the impact of using a generalized momentum distribution on the Coulomb sum rule we take as an example the simple harmonic oscillator description of ^{16}O. The associated momentum distribution, to be *ad hoc* inserted in the RFG model, reads

$$n(\eta) = \frac{8}{\sqrt{\pi}\eta_0^3}\left(1+2(\eta/\eta_0)^2\right)e^{-(\eta/\eta_0)^2} \tag{4.13}$$

with $\eta_0 = \sqrt{\omega_0/m_N}$.

In Fig. 6 and Fig. 7 we display a few versions of $\tilde{\Sigma}_C(q)$ for the momentum distribution (4.13) as obtained in the FW and scaling scheme respectively. In the left panels of the figures we have set $\eta_0 = 0.13$, which follows from the formula $\hbar\omega_0 = 41/A^{1/3}$ MeV for the harmonic oscillator frequency. In the right panels the larger value $\eta_0 = 0.2$ has instead been chosen. It is noteworthy that, in order to achieve saturation for large values of η_0, namely for extended momentum distributions, the reduction factors should be calculated utilizing (4.10) rather than (2.9).

In conclusion we provide the analytic expression for the expansion of (4.8) in the case of the momentum distribution (4.13). It reads

$$\tilde{\Sigma}_C \approx \frac{3\kappa}{2\sqrt{\pi}\eta_0} - \frac{\kappa^3}{6\sqrt{\pi}\eta_0^3} + O(\kappa^4) \tag{4.14}$$

and therefore (4.9) in this case simply becomes

$$\eta_F = \sqrt{\pi}\eta_0 \quad . \tag{4.15}$$

V. ENERGY-WEIGHTED SUM RULES

In this section we consider the energy-weighted (EWSR) and the squared-energy-weighted sum rules. For this purpose we again exploit (1.2) which allows us to express the longitudinal Pauli-blocked response function in terms of the non-Pauli-blocked one. Accordingly the EWSR may be cast in the following form

$$\begin{aligned}\Sigma_E &= \int_0^\infty d\omega\omega r_L(q,\omega) = \int_0^\infty d\omega\omega\left[r_L^{nb}(q,\omega) - r_L^{nb}(q,-\omega)\right] = \\ &= \int_{-\infty}^\infty d\omega\omega r_L^{nb}(q,\omega) \quad .\end{aligned} \tag{5.1}$$

From the above expression it follows that the passage from the Pauli-blocked regime to the non-Pauli-blocked one *leaves unaffected the functional form of* Σ_E, unlike in the case of Σ_C. More generally, the above result holds for all the sum rules with odd powers of ω.

In the non-relativistic case, where the response function is symmetric with respect to its peak, (5.1) just selects the peak itself and accordingly one has

$$\Sigma_E = \frac{q^2}{2m_N} \quad . \tag{5.2}$$

In the relativistic case, where this symmetry is lost and thus the peak of the response does not occur in correspondence with the midpoint of the response region, the EWSR is given by the relativistic extension of (5.2), namely $|Q^2|/2m_N$, plus medium dependent corrections.

To ascertain whether the latter are larger in the FW or in the scaling approach we quote the expressions for the the energy-weighted and the squared-energy-weighted sum rules (the latter only valid for $\kappa > \eta_F$) up to second order in the expansion in η_F as obtained in the scaling approach, namely [9]

$$\begin{aligned}\Xi^{(1)} &= \int_0^\infty d\omega \lambda r_L(\psi,\omega) \qquad &(5.3\text{a})\\ &= \lambda_0 + \frac{\eta_F^2}{20(1+4\kappa^2)^{3/2}}\left[1-(1+4\kappa^2)^{3/2}\right] + O\left[\eta_F^3\right]\end{aligned}$$

$$\begin{aligned}\Xi^{(2)} &= \int_0^\infty d\omega \lambda^2 r_L(\psi,\omega) \qquad &(5.3\text{b})\\ &= \lambda_0^2 + \frac{\eta_F^2}{10(1+4\kappa^2)^{3/2}}\left[(1+4\kappa^2)^{3/2} - 1 - 4\kappa^2 - 8\kappa^4\right] + O\left[\eta_F^3\right] \quad ,\end{aligned}$$

and in the FW approximation, namely

$$\begin{aligned}\Xi^{(1)} &= \int_0^\infty d\omega \lambda r_L(q,\omega) \qquad &(5.4\text{a})\\ &= \lambda_0 + \frac{3\eta_F^2}{20(1+4\kappa^2)^{3/2}}\left[1-(1+4\kappa^2)^{3/2} + \frac{8}{3}\kappa^2\right] + O\left[\eta_F^3\right]\end{aligned}$$

$$\begin{aligned}\Xi^{(2)} &= \int_0^\infty d\omega \lambda^2 r_L(q,\omega) \qquad &(5.4\text{b})\\ &= \lambda_0^2 + \frac{3\eta_F^2}{10(1+4\kappa^2)^{3/2}}\left[(1+4\kappa^2)^{3/2} - 1 - \frac{16}{3}\kappa^2 - 8\kappa^4\right] + O\left[\eta_F^3\right] \quad .\end{aligned}$$

Interestingly in both cases the variance turns out to be

$$\sigma = \sqrt{\Xi^{(2)} - (\Xi^{(1)})^2} = \frac{\kappa\eta_F}{\sqrt{5(1+4\kappa^2)}} + O\left[\eta_F^3\right] \quad . \tag{5.5}$$

We thus reach the conclusion that the FW reduced response is somewhat softened with respect to the scaling one and that both procedures lead to the same variance (or width of the reduced response at half-height) to $O\left[\eta_F^3\right]$. The softening is, however, quite modest and can only be appreciated at large momentum transfers for normal densities (see Fig. 8). It becomes pronounced at very large k_F, which of course has no physical significance, where also the widths become much different.

VI. CONCLUSIONS

In order to fulfill a relativistic Coulomb sum rule the charge response should be normalized through a factor devised to divide out the physics of the nucleon to the extent that is possible. The latter, however, is presently not calculable from a fundamental theory and is accordingly expressed through phenomenological form factors parametrized in the space-like region. Moreover the time-like physics is inaccessible with presently available experimental facilities, and hence leads us to the necessity of constructing space-like relativistic sum rules.

In this paper we have compared, in the framework of the RFG model, two different methods for setting up the normalizing factor that yields a space-like relativistic Coulomb sum rule with the same basic feature provided by the NRCSR, namely the saturation in the non-Pauli-blocked regime. The first method, established a few years ago by Alberico et al. [7], exploits the scaling behavior of the reduced longitudinal response function at large momentum transfers, whereas the second makes use of the FW transformation, which allows one to exhaust the completeness of the states of the RFG hamiltonian in the space-like domain alone.

The two methods provide reduced responses that are modestly different only at large momentum transfers (where the FW response turns out to be slightly softened with respect to the scaling one), have the same variance (up to terms proportional to η_F^2) and fulfill the Coulomb sum rule to any order in η_F in the Pauli-unblocked region. Also in the presence of Pauli blocking, the two sum rules are very close to each other in the range of k_F appropriate for nuclei, the RCSR of the FW method coinciding with the well-known non-relativistic sum rule.

We thus conclude that it is indeed possible, in the framework of the RFG, to obtain a space-like RCSR with the correct saturating behaviour for *any* k_F. Actually, as we have seen, the procedure for achieving this is not unique; but it is gratifying that the two methods we have explored yield reduced responses that are close to each other over a large span of momentum transfers at normal nuclear densities. It might be worth observing, however, that as the density and hence the magnetic contribution to the charge response (proportional to the transverse nucleon's momentum) grow, then the two reduced responses referred to above start to differ more and more. And yet the one obtained for example in the scaling approach still fulfills the sum rule at very large k_F where it becomes extremely distorted.

The question may then be asked whether a Coulomb sum rule can still be defined in the presence of strong correlations among nucleons yielding distributions extending to momenta larger than those allowed by a step-function. By inserting a simple model for such distributions in the RFG framework we have shown that the Coulomb sum rule still exists, providing the extended momentum distribution is inserted as well into the normalizing factors H_L, originally defined in terms of the pure θ-function. Whether or not our finding, obtained for a k_F corresponding to normal nuclear density, stays valid at larger k_F remains to be explored.

Furthermore, while the nucleons in the RFG are on their mass shell, they move off it when correlations come into play. We have not accounted for this physics in the present work. However we feel supported by what we view as an important finding from ref. [9] where in the framework of a hybrid model it has been shown that the structure of the normalizing factor for the RFG as provided by the scaling approach is not altered by the off-shellness of

the nucleons if an appropriate shift of the energy transfer ω is performed.

In this respect the modification of the normalizing factor introduced here to reflect the effect of an extended momentum distribution complements the one of ref. [9], where it reflects the nucleon's off-shellness. Still to be performed is a combination of the two approaches in a scheme encompassing both the nucleon's confinement (hybrid model) and realistic nucleon-nucleon correlations (extended momentum distribution).

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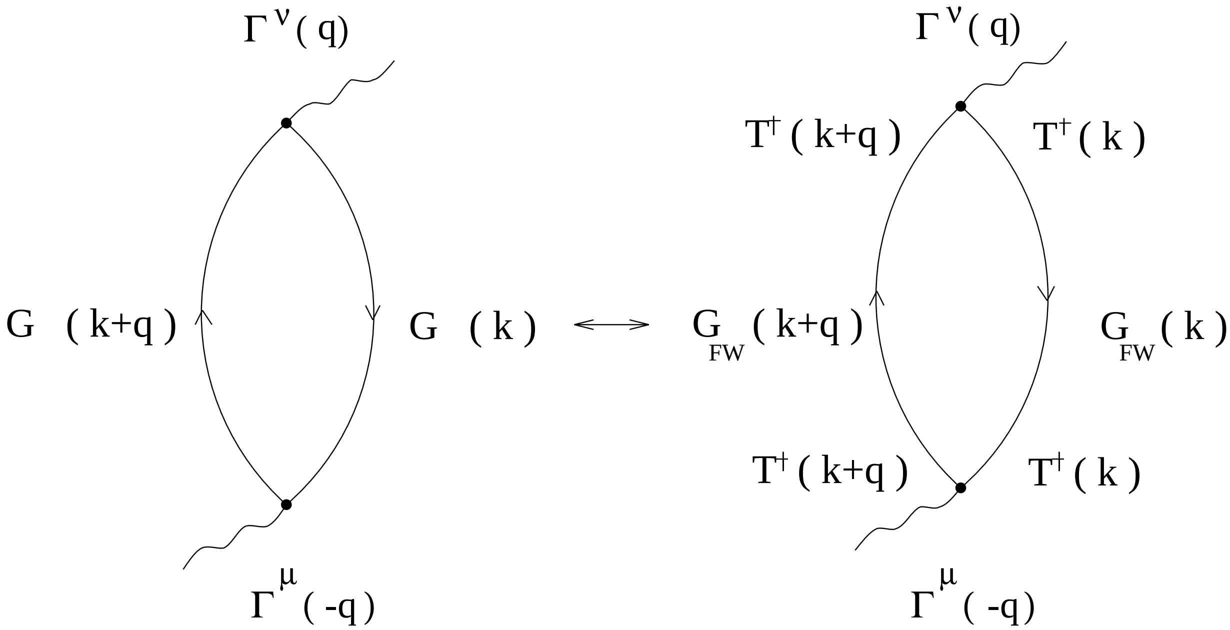

FIG. 1. Feynman's rules versus "Foldy's rules"

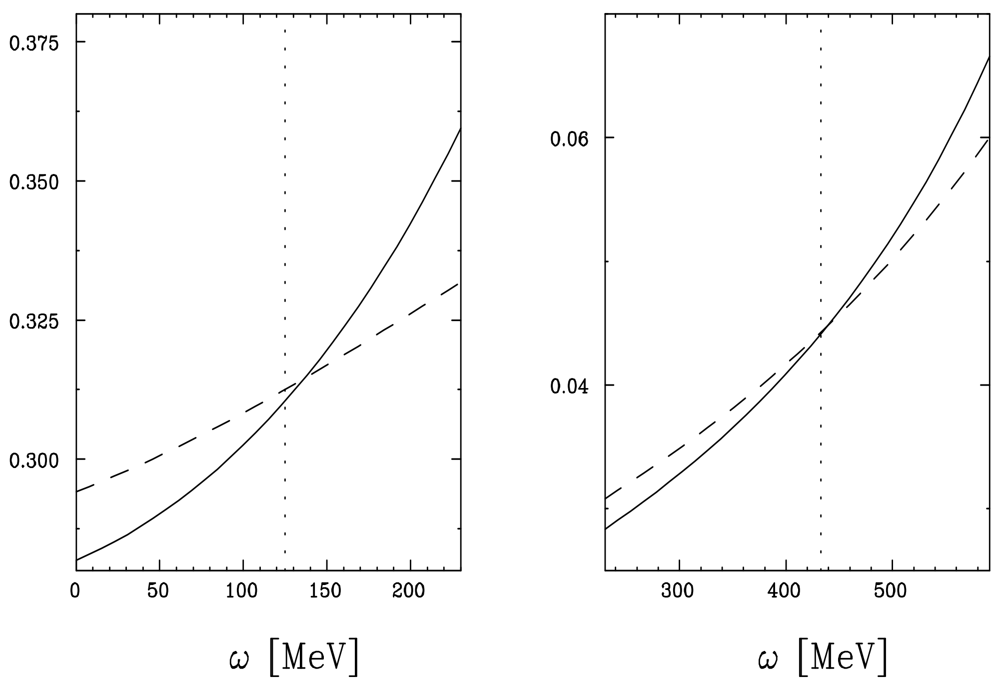

FIG. 2. H_L^{FW} (solid) and H_L^{RFG} (dashed) versus ω over the response region at $q = 500$ MeV/c (panel a) and $q = 1$ GeV/c (panel b) for a Fermi momentum of $k_F = 250$ MeV/c. The dotted line shows the position of the quasielastic peak.

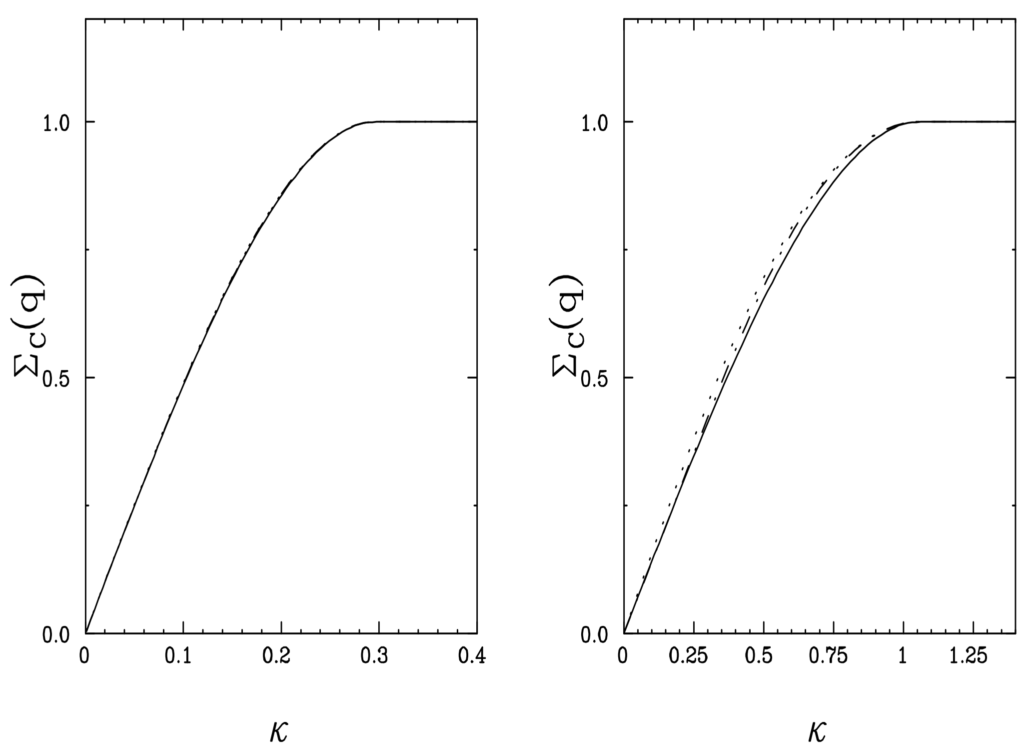

FIG. 3. Coulomb sum rule in the scaling framework: exact result numerically obtained (dot-dashed), prediction of formula (2.13) (dotted) and the NRCSR (solid). The three instances are almost indistinguishable at normal nuclear densities (left panel: $k_F = 250$ MeV/c) whereas at large density (right panel: $k_F = 1000$ MeV/c) relativity appears to mildly increase the sum rule in the non-Pauli-blocked region.

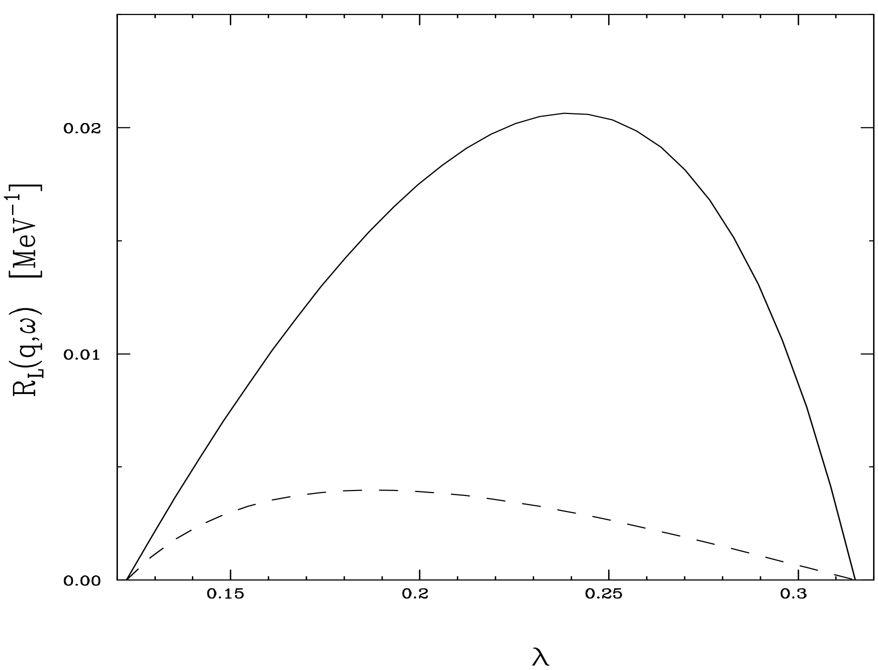

FIG. 4. Space-like (solid) and time-like (dashed) longitudinal response for Dirac nucleons versus λ, for $q = 1000$ MeV/c, $k_F = 250$ MeV/c. The time-like response region has been shifted by -ϵ_F.

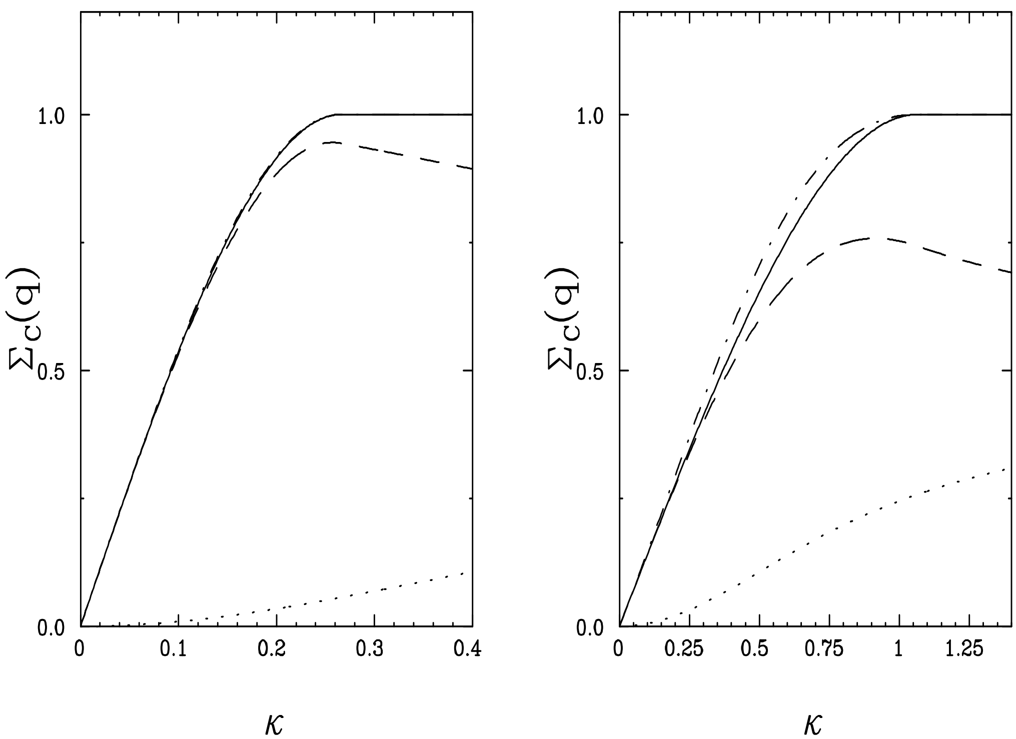

FIG. 5. Coulomb sum rule for pointlike nucleons without anomalous magnetic moments: NRCSR (solid), ph excitation (dashed), $p\overline{p}$ excitations (dotted) and $ph + p\overline{p}$ (dotdashed). Left panel: $k_F = 250$ MeV/c, right panel: $k_F = 1000$ MeV/c. Observe that only at large k_F is the difference between the NRCSR and the total relativistic sum rule perceptible. Here relativity, as in the scaling framework, appears to increase the sum rule somewhat. Observe also the growth of the $p\overline{p}$ contribution with k_F.

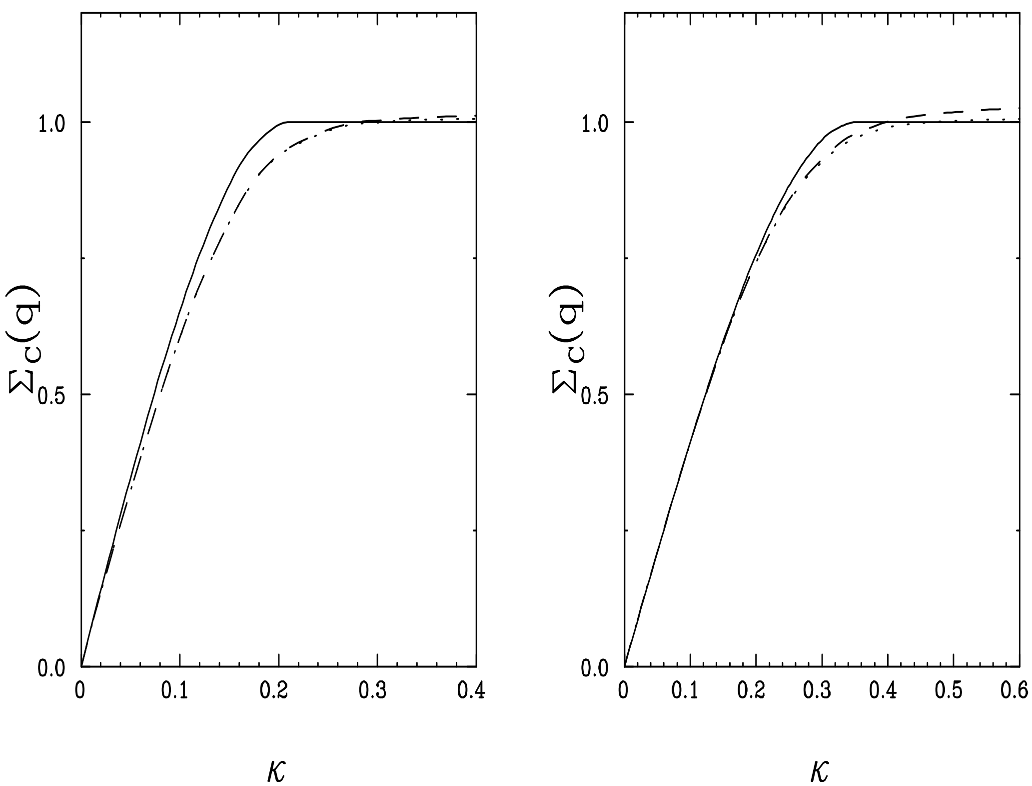

FIG. 6. The Coulomb sum rule for the RFG as given by the Foldy procedure with a shell model momentum distribution. In both panels we display the NRCSR for a Fermi momentum given by $\eta_F = \sqrt{\pi}\eta_0$ (solid line) and the sum rule corresponding to the momentum distribution (4.13) with $\eta_0 = 0.13$ (left panel) and $\eta_0 = 0.2$ (right panel). The dashed and dotted lines refer to the sum rules one obtains with the longitudinal response reduced by H_L^{FW} and by $\tilde{H}_L^{FW}$ respectively. It is evident that for large η_0 the reducing factor $\tilde{H}_L^{FW}$ is necessary to obtain the correct saturation.

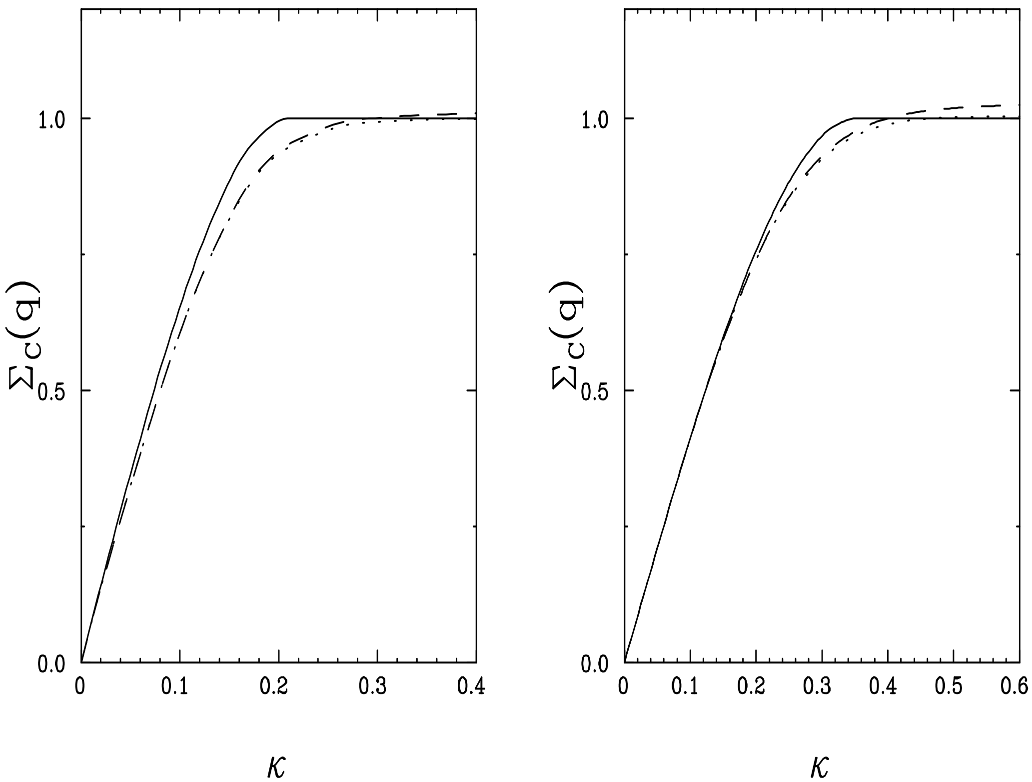

FIG. 7. The Coulomb sum rule for the RFG as obtained in the scaling framework with a shell model momentum distribution. As in the previous figure the solid line is the NRCSR for a Fermi momentum given by $\eta_F = \sqrt{\pi}\eta_0$ and the sum rule for the momentum distribution (4.13) with $\eta_0 = 0.13$ (left panel) and $\eta_0 = 0.2$ (right panel) are displayed as well. The dashed and dotted lines refer to the sum rules for the longitudinal response reduced by H_F^{FW} and by $\tilde{H}_F FW$ respectively. It is again apparent the necessity of introducing $\tilde{H}_F FW$ when η_0 is large.

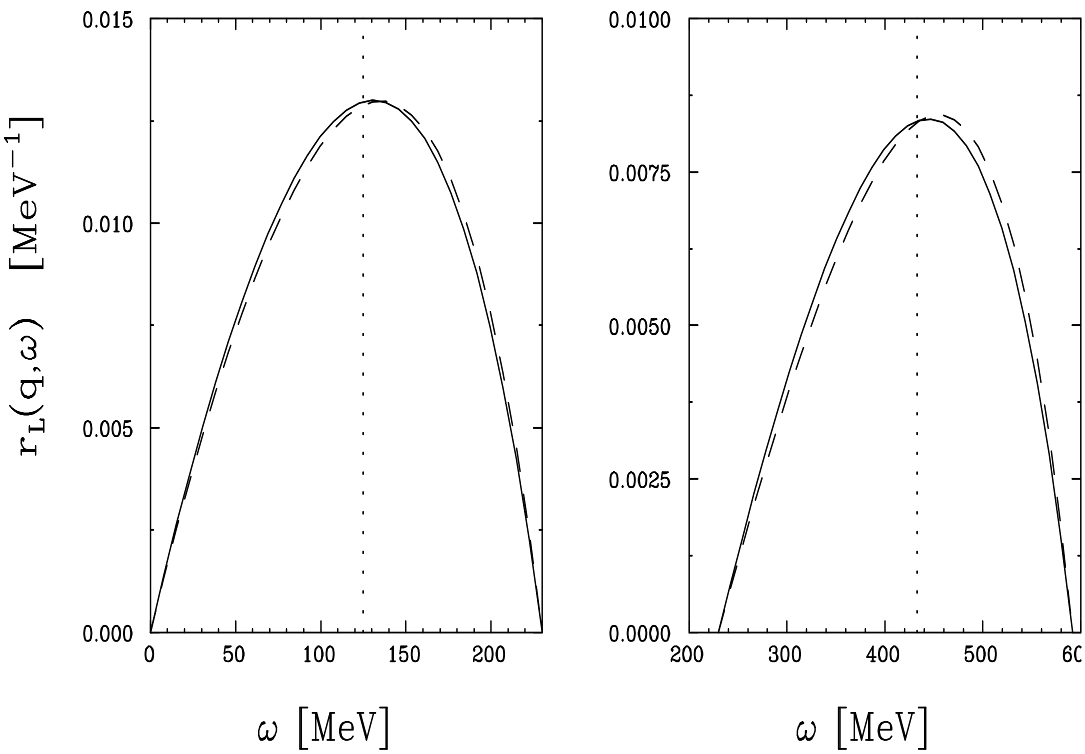

FIG. 8. The reduced responses for the FW method (solid) and for the scaling one (dashed) for $k_F = 250$ MeV/c. Left panel: $q = 500$ MeV/c, right panel: $q = 1000$ MeV/c.